Journal of Applied Mechanics and Technical Physics, Vol. 35, No. 1, 1994

OPTIMIZATION OF A LAYERED SPHERICAL INCLUSION IN AN INFINITE MATRIX WITH UNIAXIAL TENSION

V. V. Alekhin

UDC 539.3

The synthesis problem of layered bodies is one of the promising directions in the region of structural optimization. A number of studies [1-4] is devoted to them, being concerned with design problems of layered heat-protecting panels, multilayered wave filters, and elastic layered bodies. The structure and its geometric sizes are selected as control parameters in synthesis problems of layered structures. The control characterizing the structure of layered bodies is a piecewise-constant function with discrete range of values. Therefore, in deriving the control equations constructed by numerical algorithms it is necessary to use methods of the theory of optimal control. The structure and sizes of the layered construction are determined in the optimization process, though the amount, sizes, and layer materials are not known ahead of time.

In the present study we consider the synthesis problem of a finite set of elastic, homogenous, isotropic materials of multilayered spherical inclusion of minimum weight, located in a matrix stretched at infinity by a uniform uniaxial force, with given restrictions on the inclusion tensile strength and its sizes. The necessary optimization conditions have been obtained, a computational algorithm has been constructed, and a calculation example of an optimal inclusion has been provided.

1. Statement of the Problem. Let there exist a set W, consisting of k homogenous, isotropic materials. From it is required to synthesize a layered spherical inclusion of minimum weight.

Let R_1, and R_2 be the interior and exterior surface radii of the inclusion considered (see Fig. 1), located in a matrix stretched at infinity by a uniform uniaxial force q. The pressure p is assumed known at the inclusion boundary R_1. The stress-strain states of the multilayered inclusion and of the matrix are described in the case of axial symmetry by the following boundary value problem, including the equilibrium equation

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} + \sigma_{r\theta}\operatorname{ctg}\theta) = 0,$$
$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{1}{r}[3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\operatorname{ctg}\theta] = 0; \tag{1.1}$$

Hooke's law

$$\sigma_{ij} = 2G\left[\frac{\nu}{1-\nu}(e_{kl}\delta_{kl})\,\delta_{ij} + e_{ij}\right], \tag{1.2}$$

where the nonvanishing components of the strain tensor in the spherical coordinate system (r, θ, φ) are

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r},$$
$$e_{\varphi\varphi} = \frac{u_r}{r} + \frac{u_\theta}{r}\operatorname{ctg}\theta, \quad 2e_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \tag{1.3}$$

and the boundary conditions

$$\sigma_{rr}(R_1, \theta) = -p, \quad \sigma_{r\theta}(R_1, \theta) = 0,$$
$$\sigma_{rr}(\infty, \theta) = q\cos^2\theta, \quad \sigma_{r\theta}(\infty, \theta) = -q\cos\theta\cdot\sin\theta. \tag{1.4}$$

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 1, pp. 115-120, January-February, 1994. Original article submitted April 2, 1993.

0021-8944/94/3501-0117$12.50

Here $u_r(r, \theta)$, $u_\theta(r, \theta)$ are the radial and meridional displacements of body points, and $G(r)$, $\nu(r)$ are the distributed medium characteristics: the shear moduli and the Poisson coefficients of the inclusion layer materials and of the matrix.

At the internal boundaries $r_i \in (R_1, R_2)$ of the inclusion layers and at the inclusion-matrix boundary itself, where the medium properties undergo a discontinuity, it is required to assign matching conditions: continuity of displacements u_r, u_θ and stresses σ_{rr}, $\sigma_{r\theta}$, i.e.,

$$[u_r(r_i, \theta)] = [u_\theta(r_i, \theta)] = [\sigma_{rr}(r_i, \theta)] = [\sigma_{r\theta}(r_i, \theta)] = 0. \quad (1.5)$$

Let σ, L, and ρ_* be characteristic quantities, having the dimensions of stress, length, and density. Introduce new dimensionless variables (in the following we omit the asterisks of the dimensionless quantities)

$$u_i^* = u_i/L, \quad R_i^* = R_i/L, \quad \sigma_{ij}^* = \sigma_{ij}/\sigma, \quad \sigma_T^* = \sigma_T/\sigma, \quad G^* = G/\sigma, \quad p^* = p/\sigma, \quad q^* = q/\sigma, \quad \rho^* = \rho/\rho_* \quad (1.6)$$

(σ_T, ρ are the limiting stress and material density in the set W). We change the coordinates

$$r = R_1 + x(R_2 - R_1), \quad x \in [0, 1], \quad (1.7)$$

transforming the variable region $[R_1, R_2]$ into the constant [0, 1]. Introduce the piecewise-constant function

$$\alpha(x) = \{\alpha_j;\ x \in [x_j, x_{j+1}),\ j = 1, \ldots, n\}, \quad x_1 = 0, \quad x_{n+1} = 1, \quad (1.8)$$

characterizing the structure of the multilayered inclusion: the number, sizes, and composing materials of its layers. The α_j value belongs to the discrete finite set

$$U = \{\alpha_1, \ldots, \alpha_k\}, \quad (1.9)$$

corresponding to the assigned set of materials W. All the characteristics of materials of the set W are now functions of the distribution $\alpha(x)$ on the segment [0, 1]. For the set U it is convenient to assign the set of integers $U = \{1, \ldots, k\}$. The notation $\alpha(x) = i$, $x \in [x_j, x_{j+1})$ them implies that the jth spherical layer of the inclusion consists of the ith material of the set W.

Since the structure of the layered inclusion is determined by the function $\alpha(x)$, and the geometry – by its sizes R_1 and R_2, we consider the pair $\{\alpha(x), R_1\}$ as control (for definiteness the external radius R_2 is assumed fixed), where $\alpha(x) \in U$ (1.9) and

$$R_1 \in [a, b] \quad (1.10)$$

(a, b are given limits, in which one can vary the thickness of the inclusion considered).

The optimal design problem consists of the following. Among the piecewise-constant functions $\alpha(x)$ (1.8), whose range of values belongs to the set U (1.9), and the parameters R_1 of the segment $[a, b]$ (1.10) it is required to find a control $\{\alpha(x), R_1\}$, achieving a minimum of the weight functional

$$F[\alpha, R_1] = 4\pi \int_{R_1}^{R_2} \rho(\alpha) r^2 dr = \int_0^1 \Phi(x, \alpha, R_1)\, dx, \quad (1.11)$$

for given restrictions on the tensile strength

$$\eta(x, \theta, u_r, u_\theta, \sigma_{rr}, \sigma_{r\theta}, \alpha, R_1) \leq 0. \quad (1.12)$$

As a restriction (1.12) we consider the Mises flow condition

$$\eta = (\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{\varphi\varphi})^2 + (\sigma_{\varphi\varphi} - \sigma_{rr})^2 + 6\sigma_{r\theta}^2 - 2\sigma_T^2 \leq 0. \quad (1.13)$$

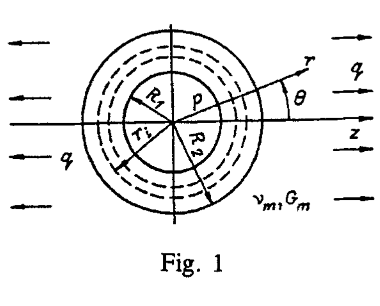

Fig. 1

Note that inequality (1.13) can be written in terms of u_r, u_θ, σ_{rr}, $\sigma_{r\theta}$, using the Hooke law relation (1.2).

2. Necessary Optimum Conditions. To derive them for the problem (1.1)-(1.13) it is required to construct an expression for the variational of the purpose functional (1.11) and the restrictions (1.13) in terms of the variation of the control $\{\alpha(x), R_1\}$. With this purpose we transform the boundary value problem (1.1)-(1.5). A solution of problem (1.1)-(1.4) was given in [5] for an arbitrary homogeneous spherical layer and for a matrix undergoing uniaxial tension by a force q at infinity.

In the layer and in the matrix the solution is

$$u_r(r,\theta) = u_{r1}(r) + u_{r2}(r)\cos 2\theta, \quad u_\theta(r,\theta) = u_{\theta 1}(r)\sin 2\theta,$$
$$\sigma_{rr}(r,\theta) = \sigma_{r1}(r) + \sigma_{r2}(r)\cos 2\theta, \quad \sigma_{r\theta}(r,\theta) = \tau(r)\sin 2\theta. \tag{2.1}$$

The stress-strain state of a matrix without inclusions, satisfying conditions (1.4) at infinity, is described by the equations

$$u_{r1} = -\frac{A_1}{r^2} - \frac{3A_2}{r^4} + \frac{5-4\nu_m}{3(1-2\nu_m)}\frac{A_3}{r^2} + \frac{q(1-\nu_m)}{4G_m(1+\nu_m)}r,$$
$$u_{r2} = -\frac{9A_2}{r^4} + \frac{5-4\nu_m}{1-2\nu_m}\frac{A_3}{r^2} + \frac{q}{4G_m}r,$$
$$u_{\theta 1} = -\frac{6A_2}{r^4} - \frac{2A_3}{r^2} - \frac{q}{4G_m}r,$$
$$\sigma_{r1} = 4G_m\left[\frac{A_1}{r^3} + \frac{6A_2}{r^5} - \frac{5-\nu_m}{3(1-2\nu_m)}\frac{A_3}{r^3}\right] + \frac{q}{2}, \tag{2.2}$$
$$\sigma_{r2} = 4G_m\left[\frac{18A_2}{r^5} - \frac{5-\nu_m}{1-2\nu_m}\frac{A_3}{r^3}\right] + \frac{q}{2},$$
$$\tau = 4G_m\left[\frac{12A_2}{r^5} - \frac{1+\nu_m}{1-2\nu_m}\frac{A_3}{r^3}\right] - \frac{q}{2}$$

(G_m, ν_m are the shear modulus and the Poisson coefficient of the matrix material).

The matching conditions (1.5) and relations (1.7), (2.1) make it possbile to to introduce continuous phase variables on the segment [0, 1]

$$\mathrm{z}(x) = (u_{r1}, u_{r2}, u_{\theta 1}, \sigma_{r1}, \sigma_{r2}, \tau)^{\mathrm{T}}. \tag{2.3}$$

Using solution (2.2) for the matrix, the original boundary value problem (1.1)-(1.4) can now be represented in the form of a boundary value problem in the unknown z (x) (2.3) for the spherical inclusion only:

$$\mathrm{z}'(x) = A(x, \alpha, R_1)\,\mathrm{z}(x), \quad z_4(0) = -p, \quad z_5(0) = z_6(0) = 0,$$
$$\mathrm{z}_f(1) = B(\nu_m, G_m, R_2)\,\mathrm{z}_l(1) + \mathrm{c}(\nu_m, G_m, R_2, q), \tag{2.4}$$

where $\mathrm{z}_f(x) = (z_1, z_2, z_3)^{\mathrm{T}}$, $\mathrm{z}_l(x) = (z_4, z_5, z_6)^{\mathrm{T}}$, while the nonvanishing elements a_{ij}, b_{ij}, c_i of the matrices $A(x, \alpha, R_1)$, $B(\nu_m, G_m, R_2)$ and of the vector $\mathrm{c}(\nu_m, G_m, R_2, q)$ are

$$a_{11} = 2a_{13} = a_{22} = \frac{2}{3}a_{23} = -a_{65} = \frac{2\nu(R_2 - R_1)}{r(\nu - 1)},$$
$$a_{14} = a_{25} = \frac{1-2\nu}{2G(1-\nu)}(R_2 - R_1), \quad \frac{1}{2}a_{32} = a_{33} = -a_{46} = -\frac{1}{3}a_{56} = -\frac{1}{3}a_{66} = \frac{R_2 - R_1}{r},$$
$$a_{36} = \frac{R_2 - R_1}{G}, \quad a_{41} = 2a_{43} = a_{52} = \frac{2}{3}a_{53} = a_{62} = 4G\frac{1+\nu}{1-\nu}\frac{R_2 - R_1}{r^2},$$
$$a_{44} = a_{55} = \frac{(2-4\nu)(R_2 - R_1)}{r(\nu - 1)}, \quad a_{63} = 2G\frac{(5+\nu)(R_2 - R_1)}{r^2(1-\nu)},$$
$$d = \frac{R_2}{8G_m(7-5\nu_m)}, \quad b_{11} = -\frac{R_2}{4G_m}, \quad b_{12} = d(3\nu_m - 1), \quad b_{13} = d(5 - 7\nu_m),$$
$$b_{22} = d(19\nu_m - 17), \quad b_{23} = d(15 - 21\nu_m), \quad b_{32} = d(10 - 14\nu_m),$$
$$b_{33} = d(26\nu_m - 22), \quad c_1 = \frac{24qd(1-\nu_m)}{1+\nu_m}, \quad c_2 = -c_3 = 30qd(1-\nu_m).$$

The local restriction (1.13) is replaced by the equivalent integral restriction

$$F_1 [z, \alpha, R_1] = \frac{1}{2} \int_V \{\eta (...) + |\eta (...)|\} dV = \int_0^1 \Phi_1 (x, z, \alpha, R_1) dx = 0. \quad (2.5)$$

Here V is the volume of the spherical inclusion, and, due to the evenness of the function $\eta(...)$ in the angle θ over the segment $[0, \pi]$, the function $\Phi_1(...)$ is

$$\Phi_1 (...) = 2\pi (R_2 - R_1) [R_1 + x (R_2 - R_1)]^2 \int_0^{\pi/2} \{\eta (...) + |\eta (...)|\} \sin \theta \, d\theta.$$

Note that the functional (2.5) is a Frechet derivative, since the integrand function $|\eta (...)|$, being the modulus of the Mises flow condition, can vanish in a layered sphere only on a set of measure zero, consisting of a finite number of points.

Let now the pair $\{\alpha (x), R_1\}$ be the optimum control of the admitted set (1.9), (1.10), minimizing the functional (1.11) and satisfying restriction (2.5). Consider the perturbed control $\{\alpha^* (x), R_1 + \delta R_1\}$ [6]

$$\alpha^* (x) = \begin{cases} g (x), & x \in D, \; g (x) \in U, \\ \alpha (x), & x \notin D, \end{cases} \qquad R_1 + \delta R_1 \in [a, b], \quad |\delta R_1| < \varepsilon \quad (2.6)$$

($D \subset [0, 1]$ is a set of low measure, mes$(D) < \varepsilon$, where $\varepsilon > 0$ is a small quantity). Using standard techniques [6], one obtains the principal portions of the increment functionals (1.11), (2.5), (for brevity we omit the arguments of the functions referring to the unperturbed control $\{\alpha (x), R_1\}$):

$$\delta F [...] = \int_D \{\Phi (\alpha^*, ...) - \Phi (\alpha, ...)\} dx + S\delta R_1,$$
$$\delta F_1 [...] = \int_D \{M (\alpha^*, ...) - M (\alpha, ...)\} dx + S_1 \delta R_1. \quad (2.7)$$

Here

$$M (x, z, \psi, \alpha, R_1) = \Phi_1 (x, z, \alpha, R_1) + \psi^T (x) A (x, \alpha, R_1) z (x);$$
$$S = \int_0^1 \frac{\partial}{\partial R_1} \Phi (x, \alpha, R_1) dx; \quad S_1 = \int_0^1 \frac{\partial}{\partial R_1} M (x, z, \psi, \alpha, R_1) dx;$$

and the vector of matched variables $\psi(x)$ satisfies the boundary value problem

$$\psi' (x) = -A^T (x, \alpha, R_1) \psi (x) - \left[\frac{\partial}{\partial z} \Phi_1 (x, z, \alpha, R_1)\right]^T,$$
$$\psi_1 (0) = \psi_2 (0) = \psi_3 (0) = 0, \quad \psi_l (1) + B^T (\nu_m, G_m, R_2) \psi_f (1) = 0. \quad (2.8)$$

We now construct the expansion functional

$$J [\alpha, R_1] = F [\alpha, R_1] + \lambda_1 F_1 [z, \alpha, R_1] + \lambda_2 \{a - R_1 + \xi_1^2\} + \lambda_3 \{R_1 - b + \xi_2^2\} \quad (2.9)$$

(λ_i, ξ_i^2, are Lagrange multipliers and penalty variables [7]). Using expression (2.7), the variational functional $J [\alpha, R_1]$ (2.9) can be represented in the form

$$\delta J [...] = \int_D \{H (\alpha, ...) - H (\alpha^*, ...)\} dx + \{S + \lambda_1 S_1 - \lambda_2 + \lambda_3\} \delta R_1 + $$
$$+ 2 (\lambda_1 \xi_1 \delta \xi_1 + \lambda_2 \xi_2 \delta \xi_2), \quad (2.10)$$

where

$$H(x, z, \psi, \alpha, R_1) = -\Phi(x, \alpha, R_1) - \lambda_1 M(x, z, \psi, \alpha, R_1). \tag{2.11}$$

Since the control $\{\alpha(x), R_1\}$ is optimal (minimal), the condition $\{\alpha^*(x), R_1 + \delta R_1\}$ must be satisfied for any admitted controls $\delta J[\ldots] \geq 0$. Due to the arbitrariness of the variations δR_1, $\delta \xi_i$, from expression (2.10) we then obtain the relations [7]

$$S + \lambda_1 S_1 - \lambda_2 + \lambda_3 = 0; \tag{2.12}$$

$$\lambda_2 (a - R_1) = 0, \quad \lambda_3 (R_1 - b) = 0, \quad \lambda_2 \geqslant 0, \quad \lambda_3 \geqslant 0 \tag{2.13}$$

and due to the fact that the set of small measure D can be distributed densely almost everywhere on the segment [0, 1], for almost all $x \in [0, 1]$ the maximum condition must be satisfied for the Hamilton function $H(\ldots)$ (2.11) in the argument α [6]:

$$H(x, z, \psi, \alpha, R_1) = \max_{\alpha^* \in U} H(x, z, \psi, \alpha^*, R_1). \tag{2.14}$$

We thus obtain that the optimal control $\{\alpha(x), R_1\}$ and the optimal trajectory corresponding to it $\mathbf{z}(x)$, as well as the vector of matched variables $\psi(x)$, must satisfy the boundary value problems (2.4), (2.8), the relations and restrictions (1.8)-(1.10), (2.5), (2.13), and the optimal conditions (2.12), (2.14).

3. Computational Algorithm. The basic idea of the direct method of solving optimal design problems consists of constructing a sequence of controls $\{\alpha(x), R_1\}_j$, $j = 1, 2, \ldots$, minimizing the purpose functional (1.11). For this we introduce a uniform grid $\{x_i\}$ by partitioning the segment [0, 1] into n segments D_i, modeling a set of small measure. We assign an initial control $\{\alpha(x), R_1\}$ from the admitted range (1.8)-(1.10), (2.5). Obviously, the function $\alpha(x)$ is piecewise constant with constant portions $D_i = [x_i, x_{i+1})$, on which it acquires values from the set U (1.9). The subsequent approximation $\{\alpha^*(x), R_1 + \delta R_1\}$ on some set D_i is sought in the form (2.6)

$$\alpha^*(x) = \begin{cases} \alpha_j, & x \in D_i, \quad \alpha_j \in U, \\ \alpha(x), & x \notin D_i; \end{cases} \tag{3.1}$$

$$R_1 + \delta R_1 \in [a, b], \quad |\delta R_1| < \varepsilon \tag{3.2}$$

and is determined from the linearized optimal problem: finding on the set D_i an admitted perturbation $\{\alpha_j, \delta R_1\}$, guaranteeing a maximum drop of the functional $F[\ldots]$ or, in different words, minimum variation of $\delta F[\ldots]$ (2.7) under conditions (3.1), (3.2) and the linearized restriction (2.5)

$$F_1[z + \delta z, \alpha^*, R_1 + \delta R_1] \approx F_1[z, \alpha, R_1] + \delta F_1[z, \alpha, R_1] = 0, \tag{3.3}$$

where the expression for $\delta F_1[\ldots]$ is given by Eq. (2.7). The given linearized problem is another version of the problem treated in Secs. 1, 2. It is hence directly obtained that the optimum perturbation $\{\alpha_j, \delta R_1\}$ must satisfy the relations

$$\delta R_1 = -\tau\{S + \lambda_1 S_1 - \lambda_2 + \lambda_3\}, \quad \tau \geqslant 0; \tag{3.4}$$

$$\lambda_2 (a - R_1 - \delta R_1) = 0, \quad \lambda_3 (R_1 + \delta R_1 - b) = 0, \quad \lambda_2 \geqslant 0, \quad \lambda_3 \geqslant 0 \tag{3.5}$$

and restrictions (3.2), (3.3).

In the computational process the factors τ, λ_2, λ_3 are found from (3.2), (3.5). The best correction α_j (3.1) is determined as follows. From relations (3.3), (3.4) one obtains

$$\delta R_1 = -\Big\{\int_{D_i} [M(\alpha_j, \ldots) - M(\alpha, \ldots)]\, dx + F_1[z, \alpha, R_1]\Big\}/S_1. \tag{3.6}$$

Minimizing the variation $\delta F[\ldots]$ (2.7), the correction α_j is then found from the condition

$$\int_{D_i} H(x, z, \psi, \alpha_j, R_1)\,dx = \max_{\alpha_* \in U} \int_{D_i} H(x, z, \psi, \alpha_*, R_1)\,dx,$$

where

$$H(x, z, \psi, \alpha_*, R_1) = -\Phi(x, \alpha_*, R_1) + SM(x, z, \psi, \alpha_*, R_1)/S_1.$$

For $S_1 = 0$ the correction obtained is determined from the relation

$$\delta R_1 = -\tau\{S - \lambda_2 + \lambda_3\}, \quad \int_{D_i} \Phi(x, \alpha_j, R_1) = \min_{\alpha_* \in U} \int_{D_i} \Phi(x, \alpha_*, R_1)\,dx$$

with account of restrictions (3.2), (3.3), (3.5).

Thus constructing the new control $\{\alpha^*(x), R_1 + \delta R_1\}$, we apply it subsequently to the initial one and construct the next approximation. The process is assumed to be finite on the given partition grid $\{x_i\}$ if the control $\{\alpha(x), R_1\}$ does not vary on any of the sets D_i. The solution obtained is a local minimum in the problem considered.

Example. The set W consists of five materials, having the following mechanical and physical dimensionless characteristics (1.6):

$$E = 270;\ 7100;\ 12000;\ 21000;\ 11200, \quad \nu = 0{,}27;\ 0{,}33;\ 0{,}32;\ 0{,}3;\ 0{,}33,$$
$$\rho = 0{,}65;\ 2{,}85;\ 4{,}6;\ 7{,}8;\ 8{,}93, \quad \sigma_T = 4{,}5;\ 44;\ 80;\ 120;\ 20$$

($E = 2G(1 + \nu)$ is the Young modulus of the material).

The pressure $p = 0$ is assigned on the internal surface of the inclusion, whose radius R_1 can vary within the limits of the segment [0.7; 0.9]. The external radius R_2 is assumed fixed and equal to unity. The matrix containing the spherical inclusion consists of the first material of the set W and is stretched at infinity by a uniaxial force $q = 4$. The inclusion region is partitioned into 50 portions of equal thickness, modeling the set D_i.

As initial approximation we selected a homogeneous inclusion of the second material with $R_1 = 0.7$. As a result of optimization we obtained a two-layered inclusion with $R_1 = 0.8992$, weight $F_* = 3.7$ and with layers [0.8992; 0.9234] of the third material, [0.9234; 1] of the second material. The lightest homogeneous inclusion, satisfying the restrictions on the tensile strength (1.13) and the body width (1.10) for given p and q, is an inclusion of the second material with $R_1 = 0.85295$ and $F^* = 4.5299$.

The relative weight advantage for the optimal inclusion in comparison with the given homogeneous one was $(1 - F_*/F^*)\cdot 100\% = 18.3\%$.

This work was supported by the grants 2-41-7-26 MNVSh and TP.

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